Random Walk Notes

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Contents

The physical set-up is a 1 dimensional chain with a random walker, the chain can be either symmetric or asymmetric, finite or infinite, equipped with various boundary condition and initial condition.

1 Group representation

Consider an infinite symmetric chain, the walker starts from the 'origin'. The recursion relation for partition function(path counting) is as follows:

$$\begin{cases} Z_{N+1}(x) = Z_N(x-1) + Z_N(x+1) \\ Z_N(x = N+1) = 0 \\ Z_{N=0} = \delta_{x,0} \end{cases}$$

We can introduce a vector which lives in l^2 space called Z_N to simplify the notation with the definition:

$$Z_N = (Z_N(1), Z_N(2), \ldots, Z_N(x), \ldots)^\top$$

In this way, the recursion relation can be expressed as a linear operator acting on a vector in l^2 space: $Z_N = TZ_{N-1}$.

Now the problem is clear, now we have a Hilbert space(Since we need completeness, inner product, orthogonal and normalized basis, then l^2 space would be convenient for us)and a linear operator. And our purpose is to solve the eigenvalue and evolution problem.

First we notice the spatial symmetry in 1D infinite symmetric chain, obviously it's $\mathbb{Z}$. And the representation for group element $+1$ and -1 should respectively corresponds to operator which moves the walker 1 step further or back. In our case we can define

$$U(+1)\,|x\rangle = |x+1\rangle$$
$$U(-1)\,|x\rangle = |x-1\rangle$$

T is the result of a group function after homomorphism, namely $T = U(+1) + U(-1)$. Also we can regard T as the representation of the generator of semi-group of Markov process. What is interesting for me is seeing that the group function can generate a semi-group, on the one hand is the spatial symmetry, on the other hand is the evolution parameter. At first I am confused since that looks like a connection between space and time. But then I realized there are two different meanings of time in physics. The first one appears in general relativity, it's a special dimension on smooth Lorentzian 4-manifold corresponding to -1 part in the signature. In this case it's a geometric object. But in other cases, time is just a name for the parameter which characterizes the evolution of system. Here the time is the latter one. So here the connection doesn't mean the connection in space-time geometry(Of course I am not talking about the connection defined on a manifold).

Since the system has spatial translational symmetry, which in the language of operator, refers to the commutation relation between U and T is $[U(n), T] = 0$. For group $\mathbb{Z}$, since it's an abelian group, it has unitary representation with no degeneracy. Since U and T commute, they share the same eigenvectors which are the Fourier modes:

$$|k\rangle = \sum_{k\in\mathbb{Z}} e^{ikx}\,|x\rangle\,, U(n)\,|k\rangle = e^{ikn}\,|k\rangle$$

The eigenvalue of T can be obtained by

$$T\,|k\rangle = (e^{ik} + e^{-ik})\,|k\rangle = 2\cos k\,|k\rangle$$

The invariant space of T is 2D here, but the additional dimension is induced by the even function $\cos k$, not by symmetry, namely it's an accidental degeneracy(not sure about that, parity perhaps).

From the transformation matrix between two basis, we obtain the common Fourier transform:

$$\sum_x Z_N(x)e^{ikx} = Z_N(k)$$

and the expression of $Z_N(k)$:

$$Z_N(k) = (2\cos k)^N Z_0(k)$$

So the workflow would be, first transform the initial condition into Fourier space, then iterate N times to get $Z_N(k)$ and later use the reverse transformation to obtain the partition function in position space.

Before proceeding into other cases or other methods, I'd like to generalize the idea above, which I think is extremely important. Of course we can regard T as group function representation, but also it's the representation of generator of Markov process. Other words, here we successfully decompose a linear operator into smaller known operators, and it is a general idea which will still survive without group structure. The following question would be, given solely a operator T, without any prior, namely in a black box, how to find the possible hidden group(if it exists)? The proper workflow should be, for a given T, first check if it's diagonalizable, if this works easily then no other methods will be needed; second, look for a family of commutants, for example here T commutes with $U(n)$, and it can be achieved by, maybe physical intuition:). So far I can only think of these two ways, but there must exists more. And also since T is the generator of semi-group, I tried to find some theorems about if there exist a unique way to decompose a semi-group generator into some group function. But unfortunately, the short answer is: no, not in full generality. Perhaps the only thing we can say is that everything is an semi-group generator problem, and symmetry is the special case where this collapses to harmonic analysis.

We can make some extension, for example, consider asymmetric transfer rate along two directions.In this case, nothing really changes, since translational invariance still holds which means the transfer operator still commute with $U(n)$. So the former choice of Fourier modes as eigenvectors is still valid. The only difference is the way of composition of T by $U(+1)$ and $U(-1)$, now it's just

$$T = aU(+1) + bU(-1)$$

in the case of

$$Z_{N+1}(x) = aZ_N(x-1) + bZ_N(x+1)$$

Thus the eigenvalues of T will also change to

$$\lambda_k = ae^{ik} + be^{-ik}$$

Also we can change the whole set-up to another group, for instance $\mathbb{Z}_{\mathbb{L}}$. The workflow will still be the same, T commute with $U(n)$, where $U(n)$ is defined as $U(n)\,|x\rangle = |x + n \mod L\rangle$. The eigenvectors for T is now discrete Fourier modes, and the eigenvalues are

$$\lambda_k = 2\cos 2\pi k$$

2 Generating function

Apparently the symmetry is quite rare and in most cases it's hard to find the group and decompose T into a group function. So we need other methods to probe the spectrum. So what does

'probe', for instance we want to probe some constants λ in a set $\mathbb{A}$, means in mathematics? A very general idea is construct some function which behave differently in these values. We have option 1, let these constants be the roots. Will that work in our case? No because that is literally solving a linear equation, which is exacting our starting point. So consider another natural idea, option 2, and construct a function whose undefined points correspond to these constants. The simplest function is the reciprocal function

$$f(x) = \frac{1}{1 - s_i x} \quad with \quad s_i = 1/\lambda$$

Now let us pass the idea in calculus to operators. For a given transfer matrix, we can regard it as a set of constants which are eigenvalues. We use exactly the same function to probe the spectrum:

$$R(s) = \frac{1}{I - sT} \quad with \quad s \in \mathbb{C}$$

Here s plays the role of 'detector', the same role as x in calculus case, but note that the eigenvalue can be complex, so s should also in complex field. The function $R(s)$ is called resolvent operator of T. If this operator is not well-defined, then that means that the operator $I - sT$ is not invertible, that is,

$$\exists \nu : (I - sT)\nu = 0$$

Then $\lambda = 1/s$ is an eigenvalue of T. See? This analogy works! It can probe the spectrum of an arbitrary matrix T. Now we entered the world of functional calculus.

(Short comments: Maybe I should change the title to Spectral Theory instead of Random Walk)

Then we can expand the resolvent operator into geometric series:

$$(I - sT)^{-1} = \sum_{n=0}^{\infty} s^n T^n, \|sT\| \leq 1$$

Multiply both sides by $Z_0(x)$, we will the familiar z-transform.

$$G(s, x) = (I - sT)^{-1} Z_0(x) = \sum_{n=0}^{\infty} s^n T^n Z_0(x) = \sum_{n=0}^{\infty} Z_N(x) s^n$$

By analogy, we can treat resolvent operator as Cauchy kernel in complex analysis. Cauchy integral formula says:

$$f(a) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - a} dz$$

then we do a replacement

$$\frac{1}{z - a} \quad \longrightarrow \quad (zI - T)^{-1}$$

Now we have the integral formula in functional calculus,

$$f(T) = \frac{1}{2\pi i} \oint_\Gamma f(z)(zI - T)^{-1} dz$$

Also we can apply a change of variable to make it look closer to the resolvent operator I mentioned before,

$$z = 1/s$$
$$dz = -\frac{1}{s^2} ds$$

then we have

$$f(T) = \frac{1}{2\pi i} \oint f(1/s)(I - sT)^{-1} \frac{ds}{s}$$

By plugging into the expression of $G(s,x)$ as a geometric series into the r.h.s formula and notice that when you integrate around zero

$$\oint_\gamma s^k ds = 0 \quad \text{unless } k = -1$$

which mean the integral is a coefficient selection machine:

$$Z_N(x) = \frac{1}{2\pi i}\oint_\gamma \frac{G(s,x)}{s^{N+1}} ds,$$

(Remark: everything above are from analogy, I just make a guess, for instance the resolvent operator, and apply a promotion from complex analysis to functional calculus. But why this promotion or analogy is valid in real mathematics, I don't know. I know they are legal but I don't know why.)

still many things are not included, I need more time later

3 Determinant Recursion Relation

still, later please

4 Quarter plane walk with small steps

4.1 Results

let me simply copy the result in paper and paste here as a reference. and actually the preliminary and notations in section 1 are only useful in section 4 and 5. I will include that later.
We restrict our attention to the quarter plane and small steps (i.e., the step set $\mathcal{S}$ is a subset of $\{-1,0,1\}^2\backslash\{(0,0)\}$). This includes the four examples above. We first narrow down the 2^8 possible cases to 79 distinct non-trivial problems (Section 2). Their step sets $\mathcal{S}$ are listed in Tables 1 to 4 in Section 8. These problems fall into two categories, depending on whether a certain group associated with $\mathcal{S}$ is finite or infinite (Section 3). The 23 models associated with a finite group turn out to be those that satisfy at least one of the following properties: - the step set possesses a vertical symmetry, - the vector sum of the vectors in the step set is 0 .

In Section 4 we develop certain general tools that apply to all models -in particular, we explain how to write for each of them a functional equation that defines the generating function of the walks. Then, we describe a uniform way to solve this equation for all of the models associated with a finite group, except one (Gessel's walks). The solutions are all D-finite, and even algebraic in three cases (Sections 5 and 6). Note that Gessel's walks are also known to have an algebraic generating function [9]. We conjecture that all the solutions to models with an infinite group are non-D-finite. We conclude in Section 7 with some comments and questions. The tables of Section 8 list the 79 models, classified according to the order of the corresponding group, and provide references to both the existing literature and to the relevant result of this paper.

4.2 Section 2 in the paper

I want to calculate numbers of cases under different conditions properly, rigorously, instead of relying on 'intuition'(Though I believe the author did it properly, but the way she presented these looks so...physical, and if I simply follow the idea in the paper I can't count properly). So let's do the classifation in 2.1 and 2.2 in a proper language.

First define the universe:

$$U = \mathcal{P}(E)$$

here E is all the 8 steps and U is all the step sets. First let us treat the symmetric bad condition properly. Define a involution and apply it on the step set:

$$\sigma : E \to E, \quad \sigma(i,j) = (j,i)$$
$$\sigma(S) = \{\sigma(e) \mid e \in S\}$$

now all the symmetric sets are:

$$\mathcal{U}^{sym} = \{S \subseteq E \mid \sigma(S) = S\}$$

now define the **condition 1** given at the beginning of 2.1:

$$E_{x+} = \{e \in E : i > 0\}$$
$$E_{x-} = \{e \in E : i < 0\}$$
$$E_{y+} = \{e \in E : j > 0\}$$
$$E_{y-} = \{e \in E : j < 0\}$$

under this condition, the collection of good sets(good steps) are:

$$\mathcal{G}_1 = \{S \subseteq E \mid S \cap E_{x+} \neq \emptyset, S \cap E_{x-} \neq \emptyset, S \cap E_{y+} \neq \emptyset, S \cap E_{y-} \neq \emptyset\}$$

and the bad collections for different conditions(bad steps) are:

$$\mathcal{B}_{11} = \{S : S \cap E_{x+} = \emptyset\}, \quad \mathcal{B}_{12} = \{S : S \cap E_{x-} = \emptyset\}, \ldots$$

and good steps can be seen as:

$$\mathcal{G}_1 = U \backslash (\mathcal{B}_{11} \cup \mathcal{B}_{12} \cup \mathcal{B}_{13} \cup \mathcal{B}_{11})$$

and here $\cup$ represents the *or* relation between conditions.

our next step is to couple the symmetric condition with condition 1 and find out good steps for both conditions, namely

$$\mathcal{G}_1^{\text{sym}} = \mathcal{U}^{\text{sym}} \cap \mathcal{G}_1$$

notice that the effect of X/Y symmetry can be seen as a reduction of condition 1 since

$$S \cap E_{x+} \neq \emptyset \iff S \cap E_{y+} \neq \emptyset$$

and $x- \iff y-$, which means the coupled condition can be simplified as:

$$\mathcal{G}_1^{sym} = \{S \in \mathcal{U}^{sym} \mid S \cap E_{x+} \neq \emptyset, S \cap E_{x-} \neq \emptyset\}$$

Now **consider condition 2**: non-negative steps. In this case we can define the condition as:

$$E_{nn} = \{(i,j) : i \geq 0, j \geq 0\}$$

and the corresponding bad steps are:

$$\mathcal{B}_2 = \{S : S \cap E_{nn} = \emptyset\}$$

we want to exclude the bad steps defined in condition 2 from $\mathcal{G}_1^{sym}$, which means we want $\mathcal{G}_2^{\text{sym}} = \mathcal{G}_1^{\text{sym}} \backslash \mathcal{B}_2$ or $\mathcal{G}_2^{\text{sym}} = \mathcal{G}_1^{\text{sym}} \setminus (\mathcal{B}_2 \cap \mathcal{G}_1^{\text{sym}})$. So we need to find the intersection $E_{x+} \cap$

$(E\backslash E_{nn}) = \{(1,-1)\}$, and symmetry gives $(-1,1)$. These two steps are forced to belong to any set in $\mathcal{B}_2 \cap \mathcal{G}_1^{\text{sym}}$.

As for condition 3, which says bad steps are those x-constrain can induce the y-constrain, or vice versa. In rigorous language, it means bad sets are:

$$\mathcal{B}_3 = \{S : \forall \text{ paths}, x \geq 0 \Rightarrow y \geq 0\}$$

and we are looking for

$$\mathcal{G}_3^{sym} = \mathcal{G}_2^{sym} \backslash (\mathcal{B}_3 \cap \mathcal{G}_2^{sym})$$

especially $\mathcal{B}_3 \cap \mathcal{G}_2^{sym}$. Since we already know that there exists $(1,-1),(-1,1)$ in $\mathcal{G}_2^{sym}$ and $(1,-1) \in \mathcal{B}_3$, so the only pair in $\mathcal{B}_3 \cap \mathcal{G}_2^{sym}$ is $\{(1,-1),(-1,1)\}$
So far what have we done? We have found the good and bad steps in three coupled conditions. Then we need to apply Burnside lemma, which is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where $X = \mathcal{G}_3$, G is the group, and $\text{Fix}(g)$ means the unchanged set under group operation. Here the group is $G = \{\text{id}, \sigma\}$. Burnside's lemma counts non-equivalent step sets up to the symmetry.

Here we consider $\mathcal{G}_3$, which can be obtained from inclusion-exclusion argument by considering condition 1,2 and 3 solely. and $\mathcal{G}_3^{\text{sym}}$ corresponds to $\text{Fix}(\sigma)$. At last we have

$$|X/G| = \frac{1}{2}(|X| + |\,\text{Fix}(\sigma)|)$$

and if we express it using generating function, that would be:

$$\frac{1}{2}\left(P_3 + P_3^{\text{sym}}\right) = 7z^3 + 23z^4 + 27z^5 + 16z^6 + 5z^7 + z^8$$

which means we have 79 non-trivial steps sets in quarter plane.

Rem1: Apparently I didn't include the detailed calculation related to generating function, but I know how to do it with IE. For example without pairs, $(1+z)$ means this point you have free choice, 1 means not to choose, z means being chosen. For example $(1+z)^8$ means all points are free, and total number of subsets are the sum of all the coefficients in this polynomials. Also if we need to select pairs, then it becomes $(1+z^2)$. And also I didn't write the explicit set elements here, you can find them in the paper. There are sentences like 'S should be the subset of something/we can only take these steps/this step is not allowed'. Also you can find these information in the examples of calculating generating functions. The idea behind this method is that, we need a mathematical structure to count. In generating functions, addition corresponds to disjoint union of combinatorial classes, while multiplication corresponds to independent choices or composition of structures whose sizes add. Then ring is the simplest structure. And polynomial ring is perfect example. But also notice that polynomial ring is commutative, what if the order of condition matters? Then very likely it will fail.

Rem2: In the last step I choose to use Burnside lemma directly instead of digging deeper as before. The reason is simple, my brain is literally burning while trying to sort different conditions and symmetry properly, like what is good sets, which part is symmetric, how to avoid over-counting, etc. So I choose to apply the lemma like a machine.

Rem3: Ist's so easy to make all kinds of mistakes in combinatorics, which is so annoying... And at some points, the logic may not be perfectly clear...

4.3 Section 3 in the paper

So at the end of chapter 2, the author has done part of the classification, now we know how many steps are non-trivial problems. But so far we only have 'geometry' of steps, so we need to convert it into something analysable. The way of doing this is using the generating polynomials, which encodes the position information of steps:

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j$$

then we can follow the paper, define the coefficients in Laurrent polynomials and find two involutions, define the induced group by composition them, which is isomorphic to dihedral group D_n of order 2n. So why they are isomorphic? Because dihedral group describes the symmetry in regular polygon with n-sides. For example, a square, it has 8 group elements, 4 reflections and 4 rotations(I treat Id as rotation). and two reflections can be treated as generators of D_4 since they can generate all the rotations in the group. And in general reflection is an example of involution, so the isomorphism here makes sense. Later the paper defines the sign of each group element in $G(\mathcal{S})$ by assigning -1 to 'reflection' and 1 to 'rotation'. Note that here the sign is not algebraic, it's just a label representing the equivalent class, since there is a homomorphism to $\{+1, -1\}$, namely the sign helps to classify 'rotation' and 'reflection', and it's always valid under group multiplication because of homomorphism. Smart idea:)

Rem1: Maybe for mathematicians, this step of converting language is normal, but it's almost magical for me. since I think the generating polynomial should carry exactly the same information as the steps set itself. But it's almost impossible to find out this group directly from steps set, but it's so clear in polynomial. Reasonable but still shocking. Of course I have seen similar examples before, like under translational symmetry, convolution can be expressed as multiplication in Fourier space. But that case it's still easy to find out the symmetry from physical set up. So how can we know that we have found out all the symmetries of an object... Maybe you can still find some using other ways to express the problem.

Later the paper show some examples generating different orders of group and also mention $G(\mathcal{S})$ is not the whole invariant group which satisfies $S(g(x, y)) = S(x, y)$, but it's the only useful group required in later discussion.(kernal invariant? not sure now)

Important result:

Theorem 3: Out of the 79 models under consideration, exactly 23 are associated with a finite group:

- 16 have a vertical symmetry and thus a group of order 4,
- 5 have a group of order 6,
- 2 have a group of order 8.

We can arrive at the theorem by...checking these steps one by one. Then we want to prove the rest of them are infinite, the proof is hard and unnecessary for me now, so I just skipped that.(Maybe I will come back later)

4.4 Section 4: general tool

The general idea here is the generating function and its kernel. Since the reasoning in the paper is so clear, I will only leave some comments here. First about the terminology itself, so far I

have seen different kinds of generating function. In some questions, people use this to encode the information of N step evolution by expanding the resolvent operator. Here more information are encoded, including the position of each trajectory. The definition is as follows:

$$Q(x,y;t) = \sum_{i,j,n\geq 0} q(i,j;n)x^i y^j t^n$$

$q(i,j;n)$ is the number of n-steps walks starting from $(0,0)$ and ending at (i,j), namely the partition function. And one step time evolution here is expressed as the product of Q and$tS(x,y)Q(x,y)$, which, in my opinion, is so smart and natural. All the physics live in polynomial rings and it's clear to process everything. Also through the discussion with Gleb, I know the generating function can be very different depending on different questions, like if we want to count how many times does the walker go up or down, we can design the generating function to do that. Though I don't know how people did this, but I guess there are many ways to do it. We can assign another variable which is not commutative just to count that, or maybe use step function to record that into the constant term.

Since the concept of kernel will be very useful in the reasoning later, let me just paste the lemma here as a definition:

Lemma 4: As a power series in t, the generating function $Q(x,y) \equiv Q(x,y;t)$ of walks with steps taken from $\mathcal{S}$ starting from $(0,0)$ and staying in the first quadrant is characterized by the following functional equation:

$$K(x,y)xyQ(x,y) = xy - txA_{-1}(x)Q(x,0) - tyB_{-1}(y)Q(0,y) + t\epsilon Q(0,0)$$

where

$$K(x,y) = 1 - tS(x,y) = 1 - t\sum_{(i,j)\in\mathcal{S}} x^i y^j$$

is called the kernel of the equation, the polynomials $A_{-1}(x)$ and $B_{-1}(y)$ are the coefficients of $\bar{y}$ and $\bar{x}$ in $S(x,y)$, and ϵ is 1 if $(-1,-1)$ is one of the allowed steps, and 0 otherwise.

And the proof of it is also a good practice for generating function I think where they concatenating walks step by step, and use inclusion-exclusion idea to add boundary terms properly.

The next important result is:
Proposition 5 (Orbit sums). Assume the group $G(\mathcal{S})$ is finite. Then

$$\sum_{g\in G} \mathrm{sign}(g)g(xyQ(x,y;t)) = \frac{1}{K(x,y;t)}\sum_{g\in G}\mathrm{sign}(g)g(xy).$$

This proposition holds because the group is finite and can be generated by involutions which will only change one variable. So when you sum over the whole orbit, you will see that some terms with opposite sign will cancel because of the sign we assigned before and the 'partly identical transformation in one variable' induced by involution(maybe there is better terminology but I don't know, I just invented this for simplicity). Because the r.h.s. is a rational function, later we will see it will give us D-finite solution in most of cases. Note that for three steps with additional vertical symmetry, $\mathcal{S}_1 = \{\bar{x},\bar{y},xy\}, \mathcal{S}_2 = \{x,y,\bar{x}\bar{y}\}$, and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, the orbit sum is trivial and the same equations can be obtained simply by vertical symmetry instead of finite group argument. Later we can solve these three using half orbit sum:
Proposition 6 (Half-orbit sums). Denote $\mathcal{S}_1 = \{\bar{x},\bar{y},xy\}$ and $\mathcal{S}_2 = \{x,y,\bar{x}\bar{y}\}$. If the set of

steps is $\mathcal{S}_1, \mathcal{S}_2$ or $\mathcal{S}_1 \cup \mathcal{S}_2$, then

$$xyQ(x,y) - \bar{x}Q(\bar{x}\bar{y},y) + \bar{y}Q(\bar{x}\bar{y},x) = \frac{xy - \bar{x} + \bar{y} - 2txA_{-1}(x)Q(x,0) + t\epsilon Q(0,0)}{K(x,y)}.$$

And about the section 4.3, the roots of kernel are(we treat it as a quartic equation w.r.t y:

$$Y_0(x) = \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)}, \quad Y_1(x) = \frac{1 - tA_0(x) + \sqrt{\Delta(x)}}{2tA_1(x)}.$$

and $Y_0(x)$ behaves well at $t \to 0$ but $Y_1(x)$ diverges. The discriminant is

$$\Delta(x) = (1 - tA_0(x))^2 - 4t^2 A_{-1}(x)A_1(x).$$

which can be factorized as:

$$\Delta(x) = \Delta_0 \Delta_-(\bar{x})\Delta_+(x)$$

here we decompose the discriminant into constant part, positive power part and negative power part.

4.5 D-Finite solutions via orbit sum

Again, let me move the result from paper to here first.
Proposition 8. For the 23 models associated with a finite group, except from the four cases $\mathcal{S} = \{\bar{x}, \bar{y}, xy\}, \mathcal{S} = \{x, y, \bar{x}\bar{y}\}, \mathcal{S} = \{x, y, \bar{x}, \bar{y}, xy, \bar{x}\bar{y}\}$ and $\mathcal{S} = \{x, \bar{x}, xy, \bar{x}\bar{y}\}$, the following holds. The rational function

$$R(x,y;t) = \frac{1}{K(x,y;t)} \sum_{g \in G} \mathrm{sign}(g) g(xy)$$

is a power series in t with coefficients in $\mathbb{Q}(x)[y, \bar{y}]$ (Laurent polynomials in y, having coefficients in $\mathbb{Q}(x)$). Moreover, the positive part in y of $R(x,y;t)$, denoted $R^+(x,y;t)$, is a power series in t with coefficients in $\mathbb{Q}[x, \bar{x}, y]$. Extracting the positive part in x of $R^+(x,y;t)$ gives $xyQ(x,y;t)$. In brief,

$$xyQ(x,y;t) = \left[x^{>}\right]\left[y^{>}\right] R(x,y;t).$$

In particular, $Q(x,y;t)$ is D-finite. The number of n-step walks ending at (i,j) is

$$q(i,j;n) = \left[x^{i+1}y^{j+1}\right] \left(\sum_{g \in G} \mathrm{sign}(g) g(xy) \right) S(x,y)^n$$

where

$$S(x,y) = \sum_{(p,q) \in \mathcal{S}} x^p y^q.$$

The proof is rather simple, the author expands the orbit sum for groups with order of 4 and extract the x-positive and y-negative part, at last only the last step of orbit will survive, which gives the general expression for generating function. And the partition function is obtained by expanding the reciprocal of kernel into geometric series and extract the coefficients. (Hey! look

familiar?) And the rest cases with a finite group are examined one by one, by hands...And time to introduce proposition 1 which is obtained from another paper to prove that it's D-finite.

Proposition 1. If $F(x, y; t)$ is a rational power series in t, with coefficients in $\mathbb{C}(x)[y, \bar{y}]$, then $[y^{>}]\, F(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. If the latter series has coefficients in $\mathbb{C}[x, \bar{x}, y]$, its positive part in x, that is, the series $[x^{>}]\, [y^{>}]\, F(x, y; t)$, is a 3 -variable D-finite series (in x, y and t).

(and later the author discusses some cases, I read them roughly, mainly these are application of orbit sum, half-orbit sum and proposition 8. Also she provides many different view points, like Young Tableaux, Motzkin number, bijection to Dyck paths, etc. I will investigate these details later, and do some calculation as practice. Also I read the paper about wetting, very very roughly, but I can see section 4.2 is almost the direct application of kernel method, and section 4.4 uses the bijection.)

5 Notes on 'Path counting on simple graphs: from escape to localization'

First I still don't understand the origin of different behavior in PC and Probability approach. Just a normalization factor? The special weight for paths going thought root in probability should should carry the same meaning in PC, where there's different numbers of path. Just normalization? Or normalization matters? But why?

In finite case, eq(9) in paper is:

$$\begin{cases} \mathrm{P}_n\left(p_0, p\right) = \lambda \mathrm{P}_{n-1}\left(p_0, p\right) - (p-1)\mathrm{P}_{n-2}\left(p_0, p\right); \\ \mathrm{P}_0\left(p_0, p\right) = \frac{p_0}{p-1}; \\ \mathrm{P}_1\left(p_0, p\right) = \lambda. \end{cases}$$

the so-called characteristic polynomial is the determinant of the matrix. This is the recursion relation for tridiagonal submatrices of A_n. The solution method can be summarized as follows: Since it's a linear recursion, or we can say it's a Markovian process, we are encouraged to use $P_n = \mu^n$ to test the solution. Insert the test solution, we obtain two possible $\mu_\pm$ and coefficient $C_\pm$. Since after substitution we will obtain a quadratic equation, by Vieta's formulas, we can reparameterize $\mu_\pm$ by $\mu_\pm = \sqrt{p-1}e^{\pm\phi}$. Plug in everything, find the real solution, take $n \to \infty$, in large N limit the biggest eigenvalue will control the behaviour of Z_N.

Now the calculation process in infinite case, which is logically deeper yet simpler for me. First let me attach the written drafts as references. And comments: Why generating function? Since the the evolution is controlled by the power of Markovian semi-group generator, which leaves us a bunch of linear equations, namely the recursion relation with step index. Then we use the resolvent operator which encodes all the information of arbitrary steps of evolution. By resolvent operator, we compress the N-dependent equations into one. But We still have spatial part in those linear equations, and we use the spatial symmetry to diagonalize this part. In this case the lattice has radial symmetry, or half line, if we consider the quotient. One important way to find the underlying group is to find the equivalent class, here the vertices with same x are in one equivalent class, thus we find the radial symmetry and corresponding sin-transformation. But in disorder system this method will be almost impossible. The group behind the model is the reason why it's exact solvable. After two transformation, we successfully compressed numerous linear equations into one 'almost algebraic' equation.

To study the phase transition, we need free energy. The sum of generating function over all possible distance corresponds to to denominator of the equation when we are calculating some

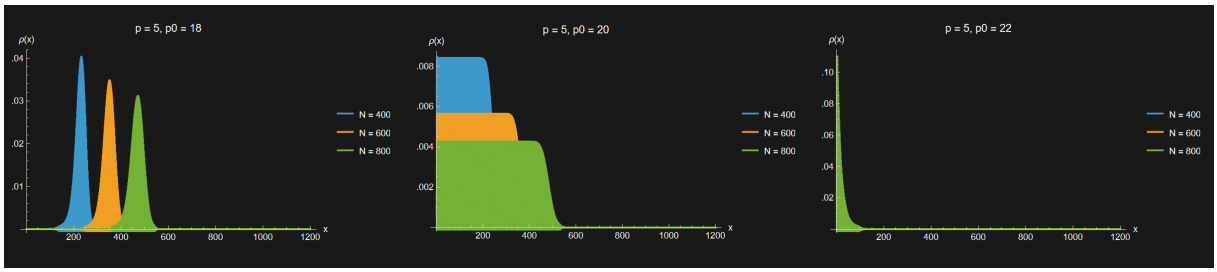

Figure 1: different phase in tree

moments, namely the normalization factor.

$$\overline{\mathcal{Z}}\left(\sigma \mid p, p_0\right) = \sum_{x=0}^{\infty} \mathcal{Z}\left(\sigma, x \mid p, p_0\right)$$

NOTE!!! in textbook, the partition function which defined the free energy has the notation Z_n, but here the same notation means the path counting function or the weight.

The singularity in generating function comes from the singularity in resolvent operator, which encodes the spectrum. the one with the least absolute value corresponds to the largest eigenvalue. (not so sure about the inheritance of singularities... one day I will need more knowledge of complex analysis)

And I like the discussion section in this paper.

6 Progress made from 19/05 to 25/05

6.1 Comparison between 'regular method' and 'one-replica' method

The number of singularities I obtained from this method is less than 3, just 1. The problem is that we infer the singularity from $R(1,t)$, instead of the full $R(x,t)$. In one-replica problem, the set up is simpler so this method works well. But the group and semi-group based full treatment is always preferred. One can try 'one-replica method' first, and compare with numerical results to probe the possible mistake.

6.2 Direct check for transition in tree graph

First I used Mathematica to get the distribution on the tree discuss in the paper and revcover the result.

And I roughly tried different regimes, like Krylov chain and linearly growing tree, and I didn't see the sign of localization. At least up to now, with several parameter I tested, all I see is Gaussian-like distribution and no phase transition. But it's too early to draw the conclusion from this.

6.3 Krylov space

I only learned the minimum amount of knowledge needed for research. In a rather practical point of view, given a Hamiltonian and a seed operator, one can obtain $\mathcal{L}^n|O\rangle$ be applying the corresponding Liouvillian operator repeatedly. And Schmidt-orthogonalization gives Lanczos

algorithm. In practice, you use $O_0 = |O\rangle, O_1 = b_1^{-1}\mathcal{L}|O\rangle$ and the recursion relation to obtain a sequence of orthogonal basis $|O_n\rangle = b_n^{-1}|K_n\rangle$, which is called Krylov basis:

$$a_n = \langle O_n|\,\mathcal{L}\,|O_n\rangle\,, \quad b_n = \langle K_n \mid K_n\rangle^{1/2}\,, \quad |K_{n+1}\rangle = (\mathcal{L} - a_n)\,|O_n\rangle - b_n\,|O_{n-1}\rangle$$

what we need is the matrix form of Liouvilian:

$$L_{nm} = \langle O_n|\,\mathcal{L}\,|O_m\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & 0 & \dots \\ b_1 & a_1 & b_2 & 0 & 0 & \\ 0 & b_2 & a_2 & b_3 & 0 & \\ 0 & 0 & b_3 & a_3 & b_4 & \\ 0 & 0 & 0 & b_4 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

So far everything looks deterministic, where is the random matrix? We treat the entries in Hamiltonian as a random variable following a Gaussian distribution, namely GOE. The resulting Liouvillian operator also has random entries. In the paper 'Matrix model for beta ensembles', a tridiagonal matrix is proved to have same statistical behavior as the original one, while in Lanczos algorithm, the same matrix is obtained by Schmidt othogonalization. At the end, we have the random matrix for Hermite case:

$$\boldsymbol{H}_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

In our problem, $\beta = 1$, the off-diagonal entries follow χ_k distribution, which has the expectation value of $\sqrt{k}$ in large k regime. So at the end of the day, we consider the expectation of the matrix and will have:

$$\begin{pmatrix} 0 & \sqrt{n-1} & 0 & \dots \\ \sqrt{n-1} & 0 & \sqrt{n-2} & \\ 0 & \sqrt{n-2} & 0 & \\ \vdots & & & \ddots \end{pmatrix}$$

6.4 Closed loop

The most important paper considered later in this section is 'Singularity analysis, Hadamard products, and tree recurrences'.

This is a tricky problem and it's easy to make mistakes. First we need to realize the partition function we got is for paths from root to equivalent class with distance x. If we want to break the loop into two N-steps path, we need to specify the ending point of the first one. I use y as an index for all the vertices. The degeneracy for each layer is

$$M_x = \begin{cases} 1, & x = 0 \\ p_0(p-1)^{x-1}, & x \geq 1 \end{cases}$$

The total number of paths from root to y after N steps is $Z_N(x)/M_x$. Since the graph is undirected, it has time reversal symmetry, which means the adjacent matrix has the property

$$A_{ij} = A_{ji}, \quad (A^N)_{ij} = (A^T)^N_{ij}.$$

Or we can say there is a bijection between the paths from 0 to y and from y to 0.

The number of all the loop is:

$$C_{2N} = \sum_y (\frac{Z_N(x)}{M_x})^2 \tag{1}$$

$$= \sum_x M_x (\frac{Z_N(x)}{M_x})^2 \tag{2}$$

$$= Z_N^2(0) + \sum_{x\geq 1} \frac{Z_N^2(x)}{p_0(p-1)^{x-1}} \tag{3}$$

Here comes the challenge. In the paper, the path of singularity transfer is simple. The singularities of $\bar{Z}_N(p, p_0)$ are from

$$\overline{\mathcal{Z}}(\sigma \mid p, p_0) = \sum_{x=0}^{\infty} \mathcal{Z}(\sigma, x \mid p, p_0)$$

Because z-transformation, sum over x, integration, all the operations are linear, so we don't need to worry about the singularity transfer. Here we have singularities of $Z_N(x)$ and are trying to obtain the singularity of

$$F(x, z) = \sum_{N=0}^{\infty} Z_N^2(x) z^N$$

First, I recognize the formula is Hadamard product which in general is defined as(We don't care about the demoninator in C_{2N} since it does't contribute to singularity):

$$a(z) \odot b(z) = \sum_{n\geqslant 0} a_n b_n z^n \quad \text{if } a(z) = \sum_{n\geqslant 0} a_n z^n, \quad b(z) = \sum_{n\geqslant 0} b_n z^n.$$

in our language,

$$a_n = b_n = Z_N(x)$$

One important theorem we would use is :

Theorem 11 (Hadamard composition of singularities). Let $f(z)$ and $g(z)$ be two functions that are Δ-regular with expansions of the type (24):

$$f(z) = \sum_{m=0}^{M} c_m (1-z)^{\alpha_m} + \mathrm{O}\left(|1-z|^A\right), \quad g(z) = \sum_{n=0}^{N} d_n (1-z)^{\beta_n} + \mathrm{O}\left(|1-z|^B\right).$$

Then, the Hadamard product $(f \odot g)(z)$ is also 4-regular. Its singular expansion is computable by bilinearity, using the composition rules of Proposition 8 and the remarks thereafter, with error terms provided by Propositions 9 and 10:

$$(f \odot g)(z) = \sum_{m,n} c_m d_n \left[(1-z)^{\alpha_m} \odot (1-z)^{\beta_n}\right] + P(1-z) + \mathrm{O}\left(|1-z|^C\right)$$

where $C := 1 + \min(\alpha_0 + B, A + \beta_0)$ and P is a polynomial of degree less than C.

and

Proposition 8. When a, b, and $a + b$ are not integers, the Hadamard product

$$(1-z)^a \odot (1-z)^b$$

has an infinite Δ-expansion with exponent scale

$$\{0,1,2,\ldots\} \cup \{a+b+1, a+b+2, \ldots\},$$

namely,

$$(1-z)^a \odot (1-z)^b \sim \sum_{k\geqslant 0} \lambda_k^{(a,b)} \frac{(1-z)^k}{k!} + \sum_{k\geqslant 0} \mu_k^{(a,b)} \frac{(1-z)^{a+b+1+k}}{k!},$$

where the coefficients λ and μ are given by

$$\lambda_k^{(a,b)} = \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)} \frac{(-a)^{\bar{k}}(-b)^{\bar{k}}}{(-a-b)^{\bar{k}}}$$
$$\mu_k^{(a,b)} = \frac{\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)} \frac{(1+a)^{\bar{k}}(1+b)^{\bar{k}}}{(2+a+b)^{\bar{k}}}$$

Here $x^{\bar{k}}$ is defined when k is a nonnegative integer as $x(x+1)\cdots(x+k-1)$.

First we rescale the generating function as

$$f(z) = \mathcal{Z}(\rho z, x) = \sum_{N=0}^{\infty} Z_N(x) \rho^N z^N$$

where ρ is one of the eigenvalue. Now the new function has the eigenvalue 1.Near $z = 1$,

$$f(z) \sim C(1-z)^\alpha$$

Then we need to decide the scaling for each eigenvalue. We only σ_1, σ_3 since σ_2 comes from the summation over x, not from the generating function.

For σ_3, it comes from the denominator thus the scaling factor $\alpha = -1$. It's consistent with the definition/property of resolvent operator/generating function. By proposition 8, the singularity comes only from $(1-z)^{2*(-1)+1}$ term. Rescale back, we obtain the corresponding eigenvalue of C_{2N} is σ_3^2.

For σ_2, it comes from the singularity of square root, not from the denominator, so we need to decide the scaling factor separately. we have

$$\sigma_1 = \frac{1}{2\sqrt{p-1}}$$

substitute into

$$\Delta(\sigma) = 1 - 4\sigma^2(p-1)$$

and we have

$$\Delta(\sigma) = 1 - \frac{\sigma^2}{\sigma_1^2} = \left(1 - \frac{\sigma}{\sigma_1}\right)\left(1 + \frac{\sigma}{\sigma_1}\right).$$

So locally,

$$\Delta(\sigma) \sim 2\left(1 - \frac{\sigma}{\sigma_1}\right).$$

At the end, the scaling factor is $\alpha = 1/2$

No, proposition 8 works only when a, b, and a + b are not integers. For my case I should use the remarks in that paper: The case where $a+b \in \mathbb{Z}$ needs transformation formula that extend (35) and are found explicitly in books by Abramowitz and Stegun [1, pp. 559-560] and by Whittaker and Watson [63, Section 14.53].

Or later I may need to check Flajolet–Odlyzko transfer theorem to obtain the coefficients

$$[z^n]\, F(z)$$

and reconstruct the F, find the singularity, solve everything.

7 Computation of closed loop

For the closed loop, the recurrence is the same:

$$\begin{cases} Z_{N+1}(0) = Z_N(1), \\ Z_{N+1}(1) = p_0 Z_N(0) + Z_N(2), \\ Z_{N+1}(x) = (p-1)Z_N(x-1) + Z_N(x+1), \quad x \geq 2, \\ Z_0(x) = \delta_{x,0} \end{cases}$$

Our goal is to find $Z_N(0)$. Now define the root-return generating function

$$G_0(\sigma) = \sum_{N \geq 0} Z_N(0)\sigma^N.$$

More generally,

$$G_x(\sigma) = \sum_{N \geq 0} Z_N(x)\sigma^N.$$

From the recurrence, we get

$$\begin{cases} G_0 - 1 = \sigma G_1, \\ G_1 = \sigma\left(p_0 G_0 + G_2\right), \\ G_x = \sigma\left((p-1)G_{x-1} + G_{x+1}\right), \quad x \geq 2. \end{cases}$$

For $x \geq 1$, use

$$G_x = C r^x.$$

Substitute into the bulk equation:

$$C r^x = \sigma\left((p-1)C r^{x-1} + C r^{x+1}\right).$$

Cancel Cr^{x-1} :

$$r = \sigma(p-1) + \sigma r^2.$$

So

$$\sigma r^2 - r + \sigma(p-1) = 0.$$

It has two branches, one with + is not physical, because if we expand it w.r.t σ, it will contain minus power term, which is impossible in our definition in generating function. The physical branch is

$$r(\sigma) = \frac{1 - \sqrt{1 - 4\sigma^2(p-1)}}{2\sigma},$$

Now use the boundary equation at $x = 1$:

$$G_1 = \sigma\left(p_0 G_0 + G_2\right).$$

Since

$$G_1 = Cr, \quad G_2 = Cr^2$$

we have

$$C\left(r - \sigma r^2\right) = \sigma p_0 G_0.$$

But from the quadratic equation,

$$r - \sigma r^2 = \sigma(p-1)$$

Therefore,

$$C = \frac{p_0}{p-1} G_0$$

Substitute into boundary condition near root,we have

$$\begin{aligned} G_0(\sigma) &= \frac{1}{1 - \frac{\sigma p_0}{p-1} r(\sigma)} \\ &= \frac{1}{1 - \frac{p_0}{2(p-1)} \left(1 - \sqrt{1 - 4\sigma^2(p-1)}\right)} \end{aligned}$$

From this expression, we can find the singularities immediately. First, one from square root:

$$\sigma_1 = \frac{1}{2\sqrt{p-1}}$$

Second, one from the denominator:

$$\sigma_3 = \frac{\sqrt{p_0 - p + 1}}{p_0}.$$

There is no $\sigma_2 = 1/p$, because σ_2 came from the open-path sum over final x. Here no such sum is performed.

8 Spectral viewpoint

Let me reformulate the idea in spectral viewpoint. Suppose A is a symmetric adjacency matrix, linear algebra tells us that there exists a set of orthogonal basis of eigenvectors

$$A\psi_\alpha = \lambda_\alpha \psi_\alpha.$$

Every vector can be expanded as

$$v = \sum_\alpha c_\alpha \psi_\alpha.$$

In discrete case, we can write the spectral decomposition as

$$A = \sum_\alpha \lambda_\alpha \left|\psi_\alpha\right\rangle \left\langle\psi_\alpha\right|$$

If we have degeneracy, suppose several orthonormal eigenvectors share the same eigenvalue λ :

$$A\psi_{\lambda,a} = \lambda \psi_{\lambda,a}, \quad a = 1, \ldots, m_\lambda.$$

Then the spectral decomposition is

$$A = \sum_\lambda \lambda P_\lambda,$$

where the projector P_λ is

$$P_\lambda = \sum_{a=1}^{m_\lambda} |\psi_{\lambda,a}\rangle \langle\psi_{\lambda,a}| .$$

Whether the matrix is degenerate or not will not affect the following content, let's just suppose it's not degenerate.

After the decomposition, power becomes trivial

$$A^N = \sum_\alpha \lambda_\alpha^N |\psi_\alpha\rangle \langle\psi_\alpha| .$$

For a graph,

$$\left(A^N\right)_{ij}$$

counts N-step walks from j to i. Using the spectral decomposition,

$$\left(\boldsymbol{A}^N\right)_{ij} = \sum_\alpha \lambda_\alpha^N \psi_\alpha(i)\psi_\alpha(j).$$

Consider the generating function, which is the entry of resolvent operator

$$\left[(I - \sigma A)^{-1}\right]_{ij} = \sum_\alpha \frac{\psi_\alpha(i)\psi_\alpha(j)}{1 - \sigma\lambda_\alpha}$$

As we can see from the formula above, in discrete spectrum, the poles of a specific generating function, in position (i, j), correspond to reciprocals of eigenvalues which can be 'seen' by the system. Being seen means that the projection of $|i\rangle, |j\rangle$ on the subspace, namely the numerator in the formula, is not zero.

Up to now, we already see something non-trivial. At first glance, we may expect to see fixed shell observable $Z_N(x)$and closed loop observable $\left(A^N\right)_{00}$ having different singularity structure since they are different objects. The spectrum seen by open paths can be obtained by summing up entries in the zeroth row in resolvent matrix which have same x, and in general $(0, 0)$ is not included. The spectrum seen by closed loops is from $(0, 0)$, so they are not necessarily the same. So I think the remarkable thing is that both fixed-shell and closed-loop seem to see the same pair σ_1, σ_3 Only the open-end observable introduces the extra σ_2, which does not belong to spectral structure of transfer matrix itself. That is not obvious a priori. Why do two completely different observables end up seeing exactly the same spectral competition?

observable	singularities visible
fixed shell $Z(\sigma, x)$	σ_1, σ_3
open path $\sum_x Z(\sigma, x)$	$\sigma_1, \sigma_2, \sigma_3$
closed loop	σ_1, σ_3

σ_1	$\frac{1}{2\sqrt{p-1}}$	edge
σ_2	$\frac{1}{p}$	sum
σ_3	$\frac{\sqrt{p_0-p+1}}{p_0}$	isolated

(I also learned a bit about spectral measure, edge of continuum, branching point, etc. But these things don't matter for now.)

I think MERW itself is not interesting in our question, they give us more detailed spectrum, and we can recover the infinite tree result by taking the limit. And we can see the bulk-edge-isolated transition in the result.

For me, another interesting question is, why does the summation over shell manufacture a new singularity?

In Burda's paper, we can see that in other cases, the eigenvalues are all bounded by $2\sqrt{p-1}$, the only possible regime is $r > 2k$, and in large G limit, the largest eigenvalue can be approximated as

$$\lambda_0 \simeq \frac{r}{\sqrt{r-k}}.$$

Now ask whether

$$\lambda_2 = k + 1$$

can be this eigenvalue. Set

$$\frac{r}{\sqrt{r-k}} = k + 1$$

The root

$$r = k + 1$$

is not in the strong-root regime $r > 2k$ for $k > 1$. The relevant solution is

$$r = k(k + 1),$$

which is exactly the transition point. So $\lambda_2 = p$ is not a generic eigenvalue of the transfer matrix. It only coincides with the root-induced eigenvalue at the open-path transition point.

9 Linearly growing tree

One example won't tell us much things, we need more examples to recoganize the common things. Now we consider the linear growing tree. The recursion relation is essentially similar, but now the coefficient has x dependence,

$$\begin{cases} Z_{N+1}(x) = xZ_N(x-1) + Z_N(x+1), & x > 2 \\ Z_{N+1}(x) = p_0 Z_N(x-1) + Z_N(x+1), & x = 1 \\ Z_{N+1}(x) = Z_N(x+1), & x = 0 \\ Z_N(x) = 0, & x \leq -1 \\ Z_{N=0}(x) = \delta_{x,0} \end{cases}$$

First, I tried to use the generating function and Fourier transform to see if it can be diagonalized. But after some work, at the end, what I obtained is not a self-consistency equation, but with a boundary term. By looking back the calculation in Cayley tree, I noticed that there's symmetrization step, which later cancels all the boundary term. At first I thought that's just for simplicity, perhaps it's necessary to arrive at the final equation. So suppose I need to factor out d_x to get the symmetric form, which means

$$Z_N(x) = d_x W_N(x)$$

in the case of $x \geq 2$, the equation becomes

$$W_{N+1}(x) = \frac{xd_{x-1}}{d_x} W_N(x-1) + \frac{d_{x+1}}{d_x} W_n(x+1)$$

Making it symmetric in matrix form means we need

$$\frac{xd_{x-1}}{d_x} = \frac{d_x}{d_{x-1}}$$

So naturally we define $d_x = \sqrt{x!}(x \geq 2)$ Following similar steps dealing with other boundary equations, at the end we have

$$d_x = \begin{cases} \sqrt{x!}, & x \geq 1 \\ \frac{1}{\sqrt{p_0}}, & x = 0 \end{cases}$$

and the corresponding adjacency matrix is

$$\begin{pmatrix} 0 & \sqrt{p_0} & 0 & 0 & 0 & \dots \\ \sqrt{p_0} & 0 & \sqrt{2} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Key observation: the bulk part is the same as harmonic oscillator and Krylov chain which can be analytically treated by Hermite function. But there is a perturbation from root, so it's a rank-2 boundary perturbation problem. As far as I know, it's a pretty standard problem in spectral theory. Maybe I can do it. Or numerically.

9.1 No perturbation

The model without perturbation (heavy root) is called 'super tree'.For a finite growing tree, the partition function satisfies the recursion

$$\begin{cases} Z_{N+1}(k) = (p_{k-1} - 1)\, Z_N(k-1) + Z_N(k+1) & \text{for } 2 \leq k \leq K-1 \\ Z_{N+1}(k) = Z_N(k+1), & \text{for } k = 0 \\ Z_{N+1}(k) = p_{k-1} Z_N(k-1) + Z_N(k+1) & \text{for } k = 1 \\ Z_{N+1}(k) = (p_{k-1} - 1)\, Z_N(k-1), & \text{for } k = K-1 \\ Z_{N=0} = \delta_{k,0} \end{cases}$$

Here p_k is the degree of freedom in k-th generation, the largest generation is K:

$$p_k = \begin{cases} p_0, & \text{for } k = 0 \\ 2 + ak, & \text{for } k \geq 1, a \geq 0 \end{cases}$$

Consider $a = 1, p_0 = 1$, the characteristic polynomials satisfy the following recursion relation:

$$\begin{cases} P_k(\lambda) = -\lambda P_{k-1}(\lambda) - (k-1) P_{k-2}(\lambda), \text{ for } 3 \leq k \leq K \\ P_1(\lambda) = \lambda \\ P_2(\lambda) = \lambda^2 - 1 \end{cases}$$

We can compare this with the recursion relation of Hermite polynomial, which is

$$\mathrm{He}_{n+1} = x\mathrm{He}_n - n\mathrm{He}_{n-1},$$

exactly the same. So the solutions are the monic Hermite polynomials.

$$P_k(\lambda) \equiv \mathcal{H}_k(\lambda) = (-1)^k e^{\frac{\lambda^2}{2}} \frac{d^k}{d\lambda^k} e^{-\frac{\lambda^2}{2}}; \quad \mathcal{H}_k(\lambda) = 2^{-k/2} H_k(\lambda/\sqrt{2})$$

Since the roots will go to infinity if we take$K \to \infty$, we need to rescale it as

$$x_i = \frac{\lambda_i}{\sqrt{K}}, \quad x_i \in [-2, 2].$$

As K goes larger, the spectral density will weakly converge (the discrete distribution will at the end 'look like' a continuous one) to Wigner semi-circle (found by Wigner in 50s, universality)

$$\rho(\lambda) = \frac{1}{2\pi K}\sqrt{4K - \lambda^2}$$

Most important conclusion: The eigenvalues of the matrix T of size $K \times K$ are the roots of the monic Hermite polynomial $\mathcal{H}_k(\lambda)$.

Remark:I found it by symmetrization, at first just to cancel the boundary term in generating function.

More or less I understand why it's related to harmonic oscillator. In Fock space we have kets $|1\rangle, |2\rangle, |3\rangle \ldots$, and creation and annihilation operator

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle.$$

Or in matrix form,

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \sqrt{3} \\ \cdots & & & \end{pmatrix}$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \cdots & & & \end{pmatrix}.$$

Since these operators satisfy the Heisenberg Algebra, so they are it's representation, also irreducible. In conclusion, Hermite polynomial is the entry of the representation of Heisenberg algebra in position base:

$$\psi_n(x) = e^{-x^2/2}H_n(x).$$

9.2 Standard treatment

First I considered the simplified problem, rank-1 perturbation. The standard treatment for a mathematician is as what follows.

Suppose first we have a known matrix J_0, we add a rank-1 perturbation with intensity g. Now the new matrix/operator looks like

$$J_g = J_0 + g\,|0\rangle\langle 0|\,.$$

We define the resolvent operator respectively:

$$\begin{aligned} R_0 &= (I - \sigma J_0)^{-1} \\ R_g &= (I - \sigma J_g)^{-1} \\ &= (I - \sigma J_0 - \sigma g\,|0\rangle\langle 0|)^{-1} \\ &= (R_0^{-1} - \sigma g\,|0\rangle\langle 0|)^{-1} \end{aligned}$$

And,

$$(R_0^{-1} - \sigma g\,|0\rangle\langle 0|)^{-1} R_g = I$$
$$R_g = \sigma g R_0\,|0\rangle\langle 0|\,R_g + R_0$$

Because we are investigating the closed loop starting from the root, we are interested in

$$\sum_N Z_N(0)\sigma^N = \langle 0| R_g |0\rangle = \sigma g \langle 0| R_0 |0\rangle \langle 0| R_g |0\rangle + \langle 0| R_0 |0\rangle$$

Define

$$\begin{cases} m_0 = \langle 0| R_0 |0\rangle \\ m_g = \langle 0| R_g |0\rangle \end{cases}$$

At the end we have

$$\boxed{m_g = \frac{m_0}{1 - \sigma g m_0}}$$

Finite case

Let

$$J_0 = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

be the spectral decomposition of J_0.

Then

$$m_0 = \left((I - \sigma J)^{-1}\right)_{00} = \sum_i \frac{\langle 0|\psi_i\rangle\langle\psi_i|0\rangle}{1 - \sigma\lambda_i}.$$

Since

$$|\psi_i(0)|^2 = \langle 0|\psi_i\rangle\langle\psi_i|0\rangle,$$

we obtain

$$m_0 = \sum_i \frac{|\psi_i(0)|^2}{1 - \sigma\lambda_i}.$$

Define the spectral weights

$$w_i := |\psi_i(0)|^2.$$

As an example, suppose

$$\lambda_1 = 1, \qquad \lambda_2 = 3,$$

with

$$w_1 = 0.4, \qquad w_2 = 0.6.$$

Then

$$m_0 = \int \frac{w_1\,\delta(\lambda - 1) + w_2\,\delta(\lambda - 3)}{1 - \sigma\lambda}\, d\lambda.$$

The corresponding spectral density is

$$\rho(\lambda) = w_1\,\delta(\lambda - 1) + w_2\,\delta(\lambda - 3).$$

Hence

$$m_0 = \int \frac{\rho(\lambda)}{1 - \sigma\lambda}\, d\lambda.$$

Infinite case

In the infinite-dimensional setting, the spectral measure may contain a continuous part. Formally,

$$m_0 = \int \frac{\rho(\lambda)}{1-\sigma\lambda}\, d\lambda,$$

where $\rho(\lambda)$ is now a continuous spectral density.

For the Hermite matrix and the vector $|0\rangle$, the spectral density is Gaussian,

$$\rho(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}.$$

Therefore

$$m_0 = \int_{-\infty}^{\infty} \frac{1}{1-\sigma\lambda} \frac{e^{-\lambda^2/2}}{\sqrt{2\pi}}\, d\lambda.$$

At the end of the day, we have

$$m_g = \frac{\int_{-\infty}^{\infty} \frac{1}{1-\sigma\lambda} \frac{e^{-\lambda^2/2}}{\sqrt{2\pi}}\, d\lambda}{1-\sigma g \int_{-\infty}^{\infty} \frac{1}{1-\sigma\lambda} \frac{e^{-\lambda^2/2}}{\sqrt{2\pi}}\, d\lambda.}$$

Now we turn to study this integral. Since there is a pole in the denominator in the integrand, it's better to change the variable to

$$E = \frac{1}{\sigma}$$

Then

$$m_0 = E M_0(E)$$

with

$$M_0(E) = \int_{-\infty}^{\infty} \frac{1}{E-\lambda} \frac{e^{-\lambda^2/2}}{\sqrt{2\pi}}\, d\lambda$$

Using Mathematica, I found the result is

$$M_0(E) = \sqrt{2} DawsonF[\frac{E}{\sqrt{2}}],$$

if E is a real.

Later, I found a note about infinite Hermite matrix, saying the spectrum is R. I am not sure, I need more time to understand that note.

Anyway, the new spectrum induced by perturbation can be found by finding the singularity in the denominator of m_g.

9.3 First return

Then I received the instruction of reading the paper about first return. I learned to obtain the partition function by divide the trajectory into loops with different revisiting time. Cool, but again, I need time.